THE EFFECT OF RADIATION ON A SMALL PARTICLE REVOLVING ABOUT JUPITER.

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1. Since the experiments of Nichols and Hull* on radiation pressure there has arisen the question of the effect of radiation in problems of celestial mechanics. Some of these problems have been already solved. Professor J. H. Poynting† has worked out the conditions of size and temperature for two bodies in equilibrium in space and also solved to a first approximation the problem of two bodies moving under the pressure of radiation and the Newtonian attraction. Professor E. B. Wilson‡ has solved the equations for this problem to a fourth approximation.

The present paper treats the motion of three bodies in one plane acting under radiation pressure as well as ordinary Newtonian forces: the third body being taken as a small satellite of Jupiter. The method of solution differs from the previous ones in that the time is not eliminated from the equations. The general effect of radiation on the particle is similar to that obtained in the above papers, namely that the particle is drawn in toward the attracting center losing eccentricity and reaching the surface after a finite number of revolutions. There are, however, some new effects. Considering the Sun to be stationary, it is clear that the particle will be in the shadow of Jupiter during a part of each revolution. It is found that the effect of the eclipse on the motion is to draw the particle in toward Jupiter and to diminish the eccentricity. But now suppose the Sun is moving. Then, if the motion of the particle is direct, the amount of shadow that the particle has to pass through will be greater than before. The particle will still be drawn in and lose eccentricity but the amount is not so great. On the other hand, if the motion of the particle is retrograde, the shadow will be less than before. Consequently the amount the particle is drawn in and the amount the eccentricity loses is greater. This shows that the number of revolutions which a particle can make with a retrograde motion is less than that which a similar particle can make with a direct motion.

2. The assumptions in regard to the radiation pressure will be the same

* "The pressure due to radiation," Astrophysical Journal, vol. 17, June, 1903.

† Nature, vol. 75, pp. 90–93. Also "Radiation in the solar system: its effect on temperature and its pressure on small bodies," Trans. Royal Society, ser. A, vol. 202 (1903).

‡ "The revolution of a dark particle about a luminous center," Annals of Math., 2d ser. vol. 7–8 (1905–07).

as in Professor Wilson's paper. That is in addition to the usual gravitational forces there is the direct pressure of light and the Doppler reception and emission effects. Besides these we shall assume the motion is all in one plane and that the particle is started well within the sphere of action of Jupiter which is taken to be one twentieth the distance from Jupiter to the Sun.

Let Jupiter be the origin of coördinates. Denote it by B and its mass by m_2. Let S be the Sun of mass m_1 and P the particle of mass m_3. Then let $BP = r$, $BS = R$, $SP = l$, $\angle PBS = \varphi$ and after Jupiter is reduced to rest let Bx be a line fixed in the plane of motion so that $\angle PBx = \vartheta$, $\angle SBx = \alpha$. Then, if V be the velocity of light, K the constant of radiation referred to S and K' that referred to B, we have for the equations of motion of the particle:*

$$\begin{aligned}\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = &-\frac{m_2}{r^2} - \frac{1}{3}\frac{K}{l^2}\frac{1}{V}\frac{dr}{dt} + \frac{m_1}{l^3}R\left(\cos\varphi - \frac{r}{R}\right) - \frac{m_1}{R^2}\cos\varphi \\ &- \frac{K}{l^3}R\left(\cos\varphi - \frac{r}{R}\right) + \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\left(\cos\varphi - \frac{r}{R}\right) \\ &+ \frac{K}{r^2} - \frac{4}{3}\frac{K'}{V}\frac{1}{r^2}\frac{dr}{dt}.\end{aligned}$$

$$\begin{aligned}\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = &-\frac{1}{3}\frac{K}{l^2}\frac{1}{V}r\frac{d\vartheta}{dt} - \frac{m_1}{l^3}R\sin\varphi + \frac{K}{l^3}R\sin\varphi \\ &- \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\sin\varphi + \frac{m_1}{R^2}\sin\varphi - \frac{1}{3}\frac{K'}{V}\frac{1}{r^2}\frac{rd\vartheta}{dt}.\end{aligned}$$

Since it is only possible to solve these equations by series and since we desire an approximation involving only the first power of the disturbance from elliptic motion, we shall divide the equations into two parts, integrate each and add the results. The division is such that the first equations, which have a potential, represent the elliptic motion of P disturbed by the attraction of the Sun and the direct pressure of light from the Sun while the second set, which has no potential, gives the motion disturbed by the Doppler effects and the direct pressure from Jupiter.

Take as the first set of equations

$$\begin{aligned}\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{m_2}{r^2} + \Bigg[&\frac{m_1}{l^3}R\left(\cos\varphi - \frac{r}{R}\right) \\ &- \frac{m_1}{R^2}\cos\varphi - \frac{K}{l^3}R\left(\cos\varphi - \frac{r}{R}\right)\Bigg]\end{aligned}$$

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = \left[\frac{m_1}{R^2}\sin\varphi - \frac{m_1}{l^3}R\sin\varphi + \frac{K}{l^3}R\sin\varphi\right].$$

* See E. B. Wilson, Annals of Math., 2d ser., vol. 7–8 (1905–07), pp. 135–137.

The right-hand sides may be expressed as partial derivatives of the function

$$F = \frac{m_2}{r} + \frac{m_1}{l} - \frac{K}{l} - \frac{m_1}{R^2} r \cos \varphi$$

$$= \frac{m_2}{r} - \frac{K}{R}\left(\frac{r}{R}\right) \cos \varphi + \frac{m_1 - K}{R}\left[\frac{1}{2}\left(\frac{r}{R}\right)^2 (3 \cos^2 \varphi - 1) + \cdots\right]$$

$$= \frac{m_2}{r} + \mathfrak{R}.$$

To solve the equations we shall develop $\mathfrak{R}$* in terms of the elliptic elements and then substitute it in the equations for the variation of the elliptic elements. These equations,† when the powers of e above the second are neglected, become

$$\frac{da}{dt} = \frac{2na^2}{m_2} \frac{\partial \mathfrak{R}}{\partial \epsilon},$$

$$\frac{de}{dt} = -\frac{nae}{2m_2} \frac{\partial \mathfrak{R}}{\partial \epsilon} - \frac{na(2 - e^2)}{2m_2 e} \frac{\partial \mathfrak{R}}{\partial \varpi},$$

$$\frac{d\varpi}{dt} = \frac{na(2 - e^2)}{2m_2 e} \frac{\partial \mathfrak{R}}{\partial e},$$

$$\frac{d\epsilon_1}{dt} = -\frac{2na^2}{m_2} \frac{\partial \mathfrak{R}}{\partial a} + \frac{nea}{2m_2} \frac{\partial \mathfrak{R}}{\partial e}.$$

After substituting and integrating with respect to the time treating the elements on the right-hand sides as constants we get the first approximation to the changes in the elliptic elements due to the disturbing forces. These are, writing the first few terms:

$$\delta a = (m_1 - K) \frac{2na^4}{m_2 a'^3} \left\{ \left[\frac{3}{2} - \frac{15}{4} e^2\right] \frac{\cos (2l + 2g - 2l')}{2(n - n')} - \frac{e}{2n} \cos l + \cdots \right\}$$

$$+ K \frac{2na^3}{m_2 a'^2} \left\{ \left[\frac{1}{2} e^2 - 1\right] \frac{\cos (l' - l - g)}{n - n'} + \frac{e}{n' - 2n} \cos (l' - 2l - g) + \cdots \right\}$$

$$= (m_1 - K) \frac{2na^4}{m_2 a'^3} \{E_1'\} + K \frac{2na^3}{m_3 a'^2} \{E_1''\}$$

say,

* Delaunay, Mém. de l'Acad. des Sci., vol. XXVIII (1860), pp. 33–54.

† Tisserand, Mécanique Céleste, vol. I, p. 169.

$$\delta e = -(m_1 - K)\frac{nea^3}{2m_2a'^3}\{E_1'\} - K\frac{nea^2}{2m_2a'^2}\{E_1''\}$$

$$-(m_1 - K)\frac{(2-e^2)na^3}{2m_2ea'^3}\left\{\frac{e}{2n}\cos l - \frac{3e}{4(3n-2n')}\cos(3l+2g-2l') + \cdots\right\}$$

$$-K\frac{(2-e^2)na^2}{2m_2ea'^2}\left\{-\frac{3e}{2n'}\cos(l-g) - \frac{e}{2(n'-2n)}\cos(l-2l-g) + \cdots\right\}$$

$$e\delta\varpi = (m_1 - K)\frac{(2-e^2)na^3}{2m_2a'^3}\left\{\frac{3}{4(3n-2n')}\sin(3l+2g-2l') - \frac{1}{2n}\sin l\right.$$

$$\left. - \frac{9}{4(n-2n')}\sin(l-2l') + \frac{3}{4}et + \cdots\right\}$$

$$+K\frac{(2-e^2)na^2}{2m_2a'^2}\left\{\frac{3}{2n'}\sin(l'-g) - \frac{1}{2(n'-2n)}\sin(l'-2l-g) + \cdots\right\}$$

$$= (m_1 - K)\frac{(2-e^2)na^3}{2m_2a'^3}\{E_2'\} + K\frac{(2-e^2)na^2}{2m_2a'^2}\{E_2''\}$$

say,

$$\delta\epsilon_1 = -(m_1 - K)\frac{4na^3}{m_2a'^3}\left\{\frac{1}{4}t + \left[\frac{3}{4} - \frac{15}{8}e^2\right]\frac{\sin(2l+2g-2l')}{2(n-n')} + \cdots\right\}$$

$$-K\frac{2na^2}{m_2a'^2}\left\{\left[\frac{1}{2}e^2 - 1\right]\frac{\sin(l'-l-g)}{n'-n} + \frac{3e}{2n'}\sin(l'-g) + \cdots\right\}$$

$$+(m_1 - K)\frac{nea^3}{2m_2a'^3}\{E_2'\} + K\frac{nea^2}{2m_2a'^2}\{E_2''\},$$

where l is the mean anomaly and g the longitude of perigee, the prime letters referring to the Sun.

We take as the second set of equations:

$$\frac{d^2r}{dt^2} - r\left(\frac{d\vartheta}{dt}\right)^2 = -\frac{m_2}{r^2} - \frac{1}{3}\frac{K}{l^2}\frac{1}{V}\frac{dr}{dt} + \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\left(\cos\varphi - \frac{r}{R}\right) + \frac{K'}{r^2} - \frac{4}{3}\frac{K'}{V}\frac{1}{r^2}\frac{dr}{dt}$$

$$= -\frac{m_2}{r^2} + P$$

say,

$$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\vartheta}{dt}\right) = -\frac{1}{3}\frac{K}{l^2}\frac{1}{V}r\frac{d\vartheta}{dt} - \frac{K}{l^3}\frac{1}{V}\frac{dl}{dt}R\sin\varphi - \frac{1}{3}\frac{K'}{V}\frac{1}{r^2}\frac{rd\vartheta}{dt}$$

$$= T$$

say.

The solution may be obtained by expanding the right-hand sides in terms of the elliptic elements and substituting in the equations for the

variation of elliptic elements which contain the forces on the right-hand sides. The expansion of dl/dt and $1/l^3$ in terms of r/R and $\cos k\varphi$ is effected as usual in the development of the perturbative function, the latter in turn being expanded in terms of the elements by use of the elliptic developments. The expansions are, to the order of e desired,

$$P = g_1\left\{\frac{na}{2}[1 + e\cos l]\sin 2\varphi - \frac{n'a}{2}[1 - e\cos l]\sin 2\varphi \right.$$
$$\left. - na[e\sin l]\cos^2\varphi\right\} - g_1\frac{na}{3}[e\sin l]$$
$$+ \frac{k}{a^2}\left[1 + 2e\cos l + \frac{1}{2}e^2 + \frac{5}{2}e^2\cos 2l\right] - 4g_2\frac{ne}{a}\sin l,$$

$$T = g_1\left\{\left[-na(1 + e\cos l) + n'a(1 - e\cos l) + \frac{na}{2}e\sin l\right]\left[\frac{1}{2} - \frac{1}{2}\cos 2\varphi\right]\right\}$$
$$- g_1\frac{na}{3}[1 + e\cos l] - g_2\frac{n}{a}[1 - 3e\cos l],$$

where $g_1 = K/R^2V$, $g_2 = K'/3V$ and where φ has yet to be expanded in terms of the elements by use of the expression

$$\varphi = \vartheta - (n't + \epsilon').$$

Neglecting higher powers the equations to be used become

$$\frac{da}{dt} = \frac{2na^2}{m_2}\left[Pae\sin\vartheta + T\frac{a^2}{r}\right],$$

$$\frac{de}{dt} = \frac{na}{m_2}\left[Pa\sin\vartheta + T\left(\frac{a^2}{r} - r - \frac{3}{2}\frac{a^2}{r}e^2 + \frac{1}{2}re^2\right)\frac{1}{e}\right],$$

$$e\frac{d\varpi}{dt} = \frac{na}{m_2}[-Pa\cos\vartheta + T(a + r)\sin\vartheta],$$

$$\frac{d\epsilon_1}{dt} = -\frac{2nar}{m_2}P.$$

In making the substitutions it will be found easier to substitute the expressions for P and T before expanding the functions of φ and ϑ into series of the elements. After integrating as before we get the first approximation to the changes due to these disturbing forces. These are, writing only the first few terms:

$$\delta a = ag_1\left[\frac{n'}{n} - 1\right]t + \frac{ag_1}{2n}\sin 2(l - l') - \frac{2ag_1e}{n}\sin l - \frac{2}{3}g_1at$$
$$- g_1\frac{4ae}{3n}\sin l - \frac{2naK'e}{m_2n}[\cos l - e\cos 2l] - g_2\frac{2t}{a} - g_2\frac{8e}{an}\sin l + \cdots,$$

$$\delta e = -\frac{5}{4}\frac{n'}{n} g_1 e t + \frac{g_1}{4}\left[e + 2 - 2\frac{n'}{n}\right]\frac{\sin(3l - 2l')}{6n - 4n'} + \frac{g_1}{n}\left[\frac{n'}{n} - 1\right]\sin l$$

$$+ \frac{g_1}{4}\left[e + 3 - 3\frac{n'}{n}\right]\frac{\sin(2l' - l)}{4n' - 2n} + \frac{g_1 e}{4}\left[\frac{n'}{n} + 2\right]\frac{\sin 2(l - l')}{2(n - n')}$$

$$- \frac{2g_1}{3n}\sin l - \frac{g_1 e}{3n}\sin 2e - \frac{K'}{m_2}\left[\left(1 + \frac{7}{8}e^2\right)\cos l + e\cos 2l\right]$$

$$- \frac{7}{2}\frac{g_2 e}{a^2} t - \frac{2g_2}{a^2 n}\sin l - \frac{5g_2 e}{4a^2 n}\sin 2l + \cdots$$

$$e\delta\varpi = \frac{g_1}{2}\left[\frac{n'}{n} - 1\right]\frac{\cos(3l - 2l')}{6n - 4n'} - \frac{3g_1}{2}\left[\frac{n'}{n} - 1\right]\frac{\cos(l - 2l')}{2n - 4n'}$$

$$- \frac{g_1}{n}\left[\frac{n'}{n} - 1\right]\cos l - \frac{2}{3}g_1 t - \frac{g_1 e}{12n}\cos l - \frac{g_1 e}{3n}\sin l$$

$$- \frac{K'}{m_2}[\sin l + e\sin 2l] + \frac{2g_2}{a^2 n}\cos l + \frac{g_2 e}{2a^2 n}\cos 2l,$$

$$\delta\epsilon_1 = \frac{g_1}{2n}\cos 2(l - l') - \frac{2g_1 e}{3n}\cos l - \frac{2nK'}{m_2}\left[t + \frac{e}{n}\sin l\right] - \frac{8g_2 e}{a^2 n}\cos l.$$

The complete integral may be obtained by adding the two sets of expressions for δa, δe, $e\delta\varpi$ and $\delta\epsilon_1$ respectively. There is a constant appearing through the mean motion which should be added to δa. This has been omitted because it does not directly concern the present problem.

3. Consider now the effect on the motion of the particle. It is evident that it will suffer solar eclipse during a portion of each revolution. During the time it is in Jupiter's shadow the radiation pressure from the Sun will be cut off. We shall first determine the effect of this eclipse on the semi major axis and eccentricity.

Denoting the angle which the intersection of the shadow and the orbit subtends at the center of Jupiter by 2β we have, when the particle enters the shadow,

$$l - l' = \pi - \beta,$$

and at the time it leaves the shadow, taking the orbit to be circular,

$$l - l' = \pi + \beta.$$

Now the change in r is given by

$$\delta r = \delta a - \delta e\cos l + e\delta\varpi\sin l,$$

neglecting products of the changes with e. From this we find the change in the value of r due to the eclipse. Since the longitude of the perijove is

constantly increasing, we can neglect, in making the substitution, all functions involving angles other than $\omega - \varpi - \omega'$. This gives us

$$\delta r = \delta a = \left[\frac{ag_1}{2n} \sin 2(l - l') \right]_{\pi+\beta}^{\pi-\beta} = -\frac{ag_1}{n} \sin 2\beta. \tag{1}$$

Hence the particle is drawn in toward the attracting center because of the eclipse.

If for an approximation we call n a constant, we have

$$\beta = \beta' \frac{1}{1 - \dfrac{n'}{n}}, \tag{2}$$

where β' is the angle between the line joining the sun and Jupiter produced, and the line joining the particle and Jupiter at the instant that the particle enters the shadow and where the particle revolves in the same direction as the Sun. If it revolves in the opposite direction,

$$\beta = \beta' \frac{1}{1 + \dfrac{n'}{n}}. \tag{3}$$

If t_1 is the time it takes the particle to make the remainder of the revolution, we have for direct motion

$$t_1 = \frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 - \dfrac{n'}{n}},$$

and

$$t_1 = \frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 + \dfrac{n'}{n}}$$

for retrograde motion.

During this time the particle is again drawn in the additional distance

$$\left[\frac{5}{3} ag_1 - ag_1 \frac{n'}{n} \right] t_1 = \frac{aK}{VR^2} \left[\frac{5}{3} - \frac{n'}{n} \right] \left[\frac{2\pi}{n} - \frac{2\beta'}{n} \frac{1}{1 - \dfrac{n'}{n}} \right] \tag{4}$$

for direct motion and

$$\frac{aK}{VR^2} \left[\frac{5}{3} - \frac{n'}{n} \right] \left[\frac{2\pi}{n} - \frac{2\beta}{n} \frac{1}{1 + \dfrac{n'}{n}} \right] \tag{5}$$

for retrograde motion.

In addition to this throughout the whole revolution the radiation from

Jupiter is affecting the particle causing it to fall in the additional distance

$$\frac{4K'\pi}{3Van}. \tag{6}$$

Thus the change in the semi major axis per revolution will be approximately

$$-\frac{aK}{VR^2n}\left\{\sin 2\beta' + 2\beta'\frac{n'}{n}\cos 2\beta' + \left[\frac{5}{3} - \frac{n'}{n}\right]\left[2\pi - 2\beta'\frac{1}{1-\frac{n'}{n}}\right]\right\} - \frac{4K'\pi}{3Van} \tag{7}$$

if the motion is direct, and

$$-\frac{aK}{VR^2n}\left\{\sin 2\beta - 2\beta'\frac{n'}{n}\cos 2\beta' + \left[\frac{5}{3} - \frac{n'}{n}\right]\left[2\pi - 2\beta'\frac{1}{1+\frac{n'}{n}}\right]\right\} - \frac{4K'\pi}{3Van} \tag{8}$$

if the motion is retrograde.

Treating similarly the expression for δe we get the change in eccentricity per revolution. This is

$$-\frac{K}{VR^2n}\left\{\frac{e}{4}\left[\frac{n'+2n}{n-n'}\right]\left[\sin 2\beta' + 2\beta'\frac{n'}{n}\cos 2\beta'\right]\right.$$
$$\left. + \frac{5e}{4}\frac{n'}{n}\left[2\pi - 2\beta'\frac{1}{1-\frac{n'}{n}}\right]\right\} - \frac{7K'e\pi}{3Va^2n}$$

when the particle revolves with the sun and

$$-\frac{K}{VR^2n}\left\{\frac{e}{4}\left[\frac{n'+2n}{n-n'}\right]\left[\sin 2\beta' - 2\beta'\frac{n'}{n}\cos 2\beta'\right]\right.$$
$$\left. + \frac{5e}{4}\frac{n'}{n}\left[2\pi - 2\beta'\frac{1}{1+\frac{n'}{n}}\right]\right\} - \frac{7K'e\pi}{3Va^2n},$$

when it revolves in a retrograde direction.

Finally the particle will not make a number of revolutions N greater than

$$N = \frac{3Vna^2R^2}{\pi\left(5Ka^2 + 4K'R^2 - 11\frac{n'}{n}Ka^2\right)} \tag{9}$$

for direct motion and

$$N_R = \frac{3Vna^2R^2}{\pi\left(5Ka^2 + 4K'R^2 + 5\frac{n'}{n}Ka^2\right)} \tag{10}$$

for the retrograde motion before it reaches the surface of Jupiter. The The difference of these will be

$$N - N_R = \frac{3Vna^2R^2 \cdot 16\frac{n'}{n}Ka^2}{\pi(5Ka^2 + 4K'R^2)\left(5Ka^2 + 4K'R^2 - 6\frac{n'}{n}Ka^2\right)}. \tag{11}$$

4. As a numerical example, we shall assume the following data.

Velocity of light,	$V = 3 \times 10^{10}$,
Mass of the Sun,	$m_1 = 1.29 \times 10^{26}$,
Mass of Jupiter,	$m_2 = 1.29 \times 10^{23}$,
Distance of Jupiter from the Sun,	$R = 7.8 \times 10^{13}$.

All of these are in the C.G.S. units.* The value of K may be determined from the known pressure of light at the earth's surface. If the particle has a density about that of the earth and a radius of about .01 cm., K has the value 1.76×10^{23}.* We also assume

Constant of radiation of Jupiter,	$K' = K \times 10^{-4}$,
Distance of the particle from Jupiter at an apse,	$a = \frac{1}{2}R \times 10^{-2}$.

Then if we suppose the shadow cast by Jupiter is a cylinder whose base is a great circle of Jupiter, β' will be equal to .0178 radians or a little more than 1°. Then it will be found from (1) with (2) or (3) that, due to the eclipse, the direct particle will fall in a distance 8.98 cm. while the retrograde particle will fall in a distance 9.40 cm. It may be noted that these values increase to a maximum when the particle has fallen in so that β' equals $\pi/4$ and then decreases to nearly 0 when the particle skims the surface of Jupiter. During the rest of the revolution equations (4), (5) and (6) show that the direct particle falls in a distance .688 km., while the retrograde particle falls in a distance .689 km. These will be approximately the totals as the eclipse effect is not quite appreciable at this distance.

Treating the expression for the change in the eccentricity in the same way we find the eclipse change for the direct particle is $.122 \times 10^{-10}e$ while that for the retrograde particle is $.117 \times 10^{-10}e$. During the rest of the revolution the change for the direct particle will be $19.15 \times 10^{-9}e$ which is approximately that for the retrograde particle. Thus the total change will be $19.16 \times 10^{-9}e$ for both particles.

To find the greatest number of revolutions we will start the particle a

* Wilson, Annals of Math., 2d ser., vol. 7–8 (1905–07), p. 148.

little farther out from Jupiter and let

$$\text{Distance from Jupiter, } a = R \times 10^{-2}.$$

Then the number of revolutions which is found from (9) or (10) probably will not be more than 5.87×10^7 for the direct and 5.54×10^7 for the retrograde particle. The difference between these, given by (11), is 3.3×10^6. This indicates that the retrograde particle is drawn in much faster, a fact which can be verified in general by comparing equations (7) and (8).